Two-Mode Nonlinear Vibration of Orthotropic Plates Using Method of Multiple Scales

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Nonlinear forced oscillation of a rectangular orthotropic plate subjected to uniform harmonic excitation is solved using the method of multiple scales. The governing equations are based on the von Kármán type geometrical nonlinearity, and the effect of damping is included. The general multimode solution is developed for simply supported boundary conditions, and the solution is specialized for two-symmetric modes analysis. The primary resonances and the subharmonic and superharmonic secondary resonances are studied in detail.

Introduction

THIN laminated composite panels subjected to transverse periodic loadings can encounter deflections of the order of panel thickness or even higher. The effect of these periodic excitations on the panel can be very severe. Responses of this kind cannot be predicted by linear theory. Consequently, the need to study large deflections using nonlinear methods of analysis is of paramount importance.

The formulation of the equations governing the fundamental kinematic behavior of the laminated composite plates in the presence of the von Kármán geometrical nonlinearity is attributed to Whitney and Leissa.[1] Based on these equations, various methods have been developed to solve nonlinear free and forced vibrations of composite panels. A good survey on mainly nonlinear free and forced vibrations of isotropic plates is given in a book by Nayfeh and Mook.[2] The most comprehensive work on geometrically nonlinear analysis of both static and dynamic behavior of the laminated panels through 1972 is collected in a book by Chia.[3] Bert[4] has conducted a survey on the dynamics of composite panels for the period of 1979–81. A review of literature on linear vibrations of plates can be found in a review paper by Sathyamoorthy.[5]

Relatively few investigations have been reported on the nonlinear forced vibration of isotropic or composite panels under harmonic excitations. Yamaki[6] presented a one-term solution for free and forced vibrations of the rectangular plates, using Galerkin's method. Lin[7] studied the response of a nonlinear flat panel to periodic and randomly varying loadings. Nonlinear forced vibrations of beams and rectangular plates were studied by Eisely[8] using a single-mode Galerkin's method in conjunction with the Linstedt-Duffing perturbation technique. Free and forced response of beams and plates undergoing large-amplitude oscillations using the Ritz averaging method were studied by Srinivasan.[9] Bennett[10] studied the nonlinear vibration of simply supported angle-ply laminated plates by considering the instability regions of the response of such plates subjected to harmonic excitations. Nonlinear free and forced vibration of a circular plate with clamped boundary conditions using the Kantorovich averaging method was studied by Huang and Sandman.[11] Kung and Pao[12] solved the nonlinear response of a circular clamped panel to harmonic excitation using Galerkin's method. The nonlinear free and forced vibration of a clamped orthotropic circular plate with a concentric core of isotropic material was studied by Huang[13] using the Ritz-Kantorovich method in conjunction with Newton's integration method. Rehfield[14] studied the large-amplitude forced vibration of beams and plates using perturbation methods. Free and forced vibration of an axisymmetric circular plate with various boundary conditions using the Kantorovich averaging technique was studied by Huang and Al-Khattat.[15] Yamaki, et al.[16] studied the nonlinear forced vibration of a clamped circular plate with initial deflection and initial edge displacement by using a three-term Galerkin method in conjunction with the harmonic balance method. Large-deflection static bending and large-amplitude free, randomly forced and harmonically forced vibrations of a clamped symmetrically laminated rectangular plate were studied by Gary et al.[17] Recently, Mei and Decha-Umphai[18] developed a finite-element method to analyze the forced vibration of isotropic rectangular plates without the effect of damping. Decha-Umphai[19] studied the forced vibration of circular plates by using the finite-element method. This method also is used to solve the nonlinear forced vibration of symmetrically laminated rectangular plates by Mei and Chiang.[20] Later, Wentz et al.[21] presented the forced vibration response of a generally laminated rectangular plate.

Various perturbation techniques have been used to analyze free and forced vibration of plates. The method of multiple scales is one of the most widely used because of its versatility in dealing with nonlinear structural problems. For example, Crawford and Atluri[22] investigated the nonlinear free vibration of a flat plate with initial stresses by using a multimode, multiple-scales method. Nonlinear forced responses of symmetric as well as asymmetric circular plates were studied by Sridhar et al.[23,24] This method also was used to treat elliptical and irregular plates by Lobitz et al.[25]

There is no literature available on the nonlinear forced vibrations of orthotropic plates with the effect of damping for all the possible resonances other than the recent paper by Eslami and Kandil,[26] which deals with the single-mode analysis. The purpose of the present paper is to study the nonlinear forced oscillations of an orthotropic panel subjected to harmonic excitation for a two-mode analysis. The Galerkin method is used to transform the nonlinear partial differential equations into a set of nonlinear ordinary differential equations. The resulting equations are solved by employing the method of multiple scales. By using the method of multiple

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scales, subharmonic and superharmonic resonances, as well as primary resonances, are studied.

Formulation

The governing differential equations for a rectangular, orthotropic panel subjected to harmonic loading $E(t)$, including the effect of viscous damping and in-plane loadings, are given by

$$\rho\, h\ddot{w} + g\,\dot{w} + D_{11}\, w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy}$$
$$- [(\Phi_{,yy} + N_x^a)w_{,xx} + (\Phi_{,yy} + N_y^a)\, w_{,yy}$$
$$- 2\,\Phi_{,xy}\, w_{,xy}] - E(t) = 0 \tag{1}$$

$$A_{22}^*\,\Phi_{,xxxx} + (2A_{12}^* + A_{66}^*)\,\Phi_{,xxyy} + A_{11}^*\,\Phi_{,yyyy}$$
$$= w_{,xy}^2 - w_{,xx}\, w_{,yy} \tag{2}$$

In Eqs. (1) and (2), A_{ij}^* and D_{ij} are the laminate extensional and bending stiffnesses, N_x^a and N_y^a the inplane loads, w the panel deflection, Φ the Airy stress function, ρ the mass density, h the panel thickness, and g the damping coefficient. The formulation is focused on a uniformly distributed harmonic loading described by

$$E(t) = E_0 \cos\omega t \tag{3}$$

Carrying out the procedure described in Ref. 26 and introducing the following parameters

$$\Omega = (\rho h\, a_0^4/D_{22})^{1/2}\omega \tag{4a}$$

$$P_0 = (E_0\, a_0^4/h\, D_{22}) \tag{4b}$$

$$C = [g(\rho_m h)^{-1/2} a_0^4/D_{22}^{1/2}] \tag{4c}$$

the following nondimensional equations are obtained:

$$L(\bar{w},F) = \bar{w}_{,\tau\tau} + C\,\bar{w}_{,\tau} + 2D^*\, r_0^2\bar{w}_{,\xi\xi\eta\eta} + r_0^4\bar{w}_{,\eta\eta\eta\eta}$$
$$- r_0^2[(F_{,\eta\eta} + N_\xi^a)\,\bar{w}_{,\xi\xi} + (F_{,\xi\xi} + N_\eta^a)\,\bar{w}_{,\eta\eta}$$
$$- 2\,F_{,\xi\eta}\,\bar{w}_{,\xi\eta}] - P_0\cos\Omega\tau = 0 \tag{5}$$

$$\alpha\, F_{,\xi\xi\xi\xi} + \beta\, r^2\, F_{,\xi\xi\eta\eta} + \gamma\, r^4\, F_{,\eta\eta\eta\eta} = r^2\,(\bar{w}^2_{,\xi\eta} - \bar{w}_{,\xi\xi}\,\bar{w}_{,\eta\eta}) \tag{6}$$

All the dimensionless quantities in Eqs. (5) and (6) are defined in Ref. 26. Galerkin's method is applied to reduce the set of partial differential equations to a set of ordinary differential equations by assuming a solution of the form

$$\bar{w} = \sum_m\sum_n q_{mn}(\tau)\, X_m(\xi)\, Y_n(\eta) \tag{7}$$

Substituting Eq. (7) into Eq. (6) gives an equation of the form

$$\alpha F_{,\xi\xi\xi\xi} + \beta r^2\, F_{,\xi\xi\eta\eta} + \gamma r^4 F_{,\eta\eta\eta\eta}$$
$$= r^2 \sum_m\sum_t\sum_n\sum_s q_{mt}\, q_{ns}\,(x_m'\, x_n'\, y_t'\, y_s' - x_m''\, x_n\, y_t''\, y_s) \tag{8}$$

where

$$x' = \frac{\mathrm{d}x}{\mathrm{d}\xi},\; y' = \frac{\mathrm{d}y}{\mathrm{d}\eta}$$

Solution of this linear partial differential equation may be written as

$$F(\xi,\eta,\tau) = F^h(\xi,\eta,\tau) + F^p(\xi,\eta,\tau) \tag{9}$$

in which F^h, the homogeneous solution, includes the contribution from the inplane boundary conditions and F^p, the particular solution, includes the contribution from the out-of-plane boundary condition. The homogeneous solution can be determined by assuming

$$F^h(\xi,\eta,\tau) = \bar{N}_\xi(\eta^2/2) + \bar{N}_\eta(\xi^2/2) \tag{10}$$

Using Eq. (10) along with the in-plane boundary conditions[26] and performing the integrations, we obtain

$$\begin{bmatrix} r^2\gamma & \beta_1 \\ r^2\beta_1 & \alpha \end{bmatrix} \begin{Bmatrix} \bar{N}_\xi \\ \bar{N}_\xi \end{Bmatrix} = \begin{Bmatrix} I_\xi \\ I_\eta \end{Bmatrix} \tag{11}$$

where

$$I_\xi = \frac{1}{2}\int_0^1\int_0^1 \bar{w}_{,\xi}^2\,\mathrm{d}\xi\,\mathrm{d}y$$

$$I_\eta = \frac{r^2}{2}\int_0^1\int_0^1 \bar{w}_{,\eta}^2\,\mathrm{d}\xi\,\mathrm{d}y \tag{12}$$

This solution corresponds to the in-plane boundary conditions in which the edges are restrained from any movement.

To obtain the particular solution F_p, we assume a solution of the form

$$F_{mtns}^p = \sum_m\sum_t\sum_n\sum_s q_{mt}\, q_{ns}\,\bar{F}_{mnts}(\xi,\eta) \tag{13}$$

in which $\bar{F}_{mnts}$ is a solution to

$$\alpha\,\bar{F}_{,\xi\xi\xi\xi} + \beta r^2\,\bar{F}_{,\xi\xi\eta\eta} + \gamma r^4\,\bar{F}_{,\eta\eta\eta\eta}$$
$$= r^2(x_m'\, x_n'\, y_t'\, y_s' - x_m''\, x_n\, y_t''\, y_s) \tag{14}$$

Once the total solution for F is obtained, then by substituting $\bar{w}$ and F into the governing differential equation $L(\bar{w},F) = 0$, making use of the admissible function as a weighted function and setting the weighted residual equal to zero, i.e.,

$$\int_0^1\int_0^1 L(\bar{w},F)\, x_\ell(\xi)\, y_j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0 \tag{15}$$

a set of nonlinear ordinary differential equations in terms of q is obtained

$$\sum_\ell\sum_j (M_{k\ell}^{pj}\,\ddot{q}_{\ell j} + C_{k\ell}^{pj}\,\dot{q}_{\ell j} + K_{k\ell}^{pj}\, q_{\ell j})$$
$$+ \sum_\ell\sum_j\sum_m\sum_t\sum_n\sum_s B_{k\ell mn}^{pjts}\, q_{\ell j}\, q_{mt}\, q_{ns} = P_{kp}\cos\Omega\tau \tag{16}$$

where

$$P_{kp} = \int_0^1\int_0^1 P_0\, X_k(\xi)\, Y_p(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{17}$$

This equation can be written as a system of $M \times N$, coupled nonlinear differential equations in time for the unknown q_{mn} in the form

$$\ddot{\tilde{q}}_k + c_k\,\dot{\tilde{q}}_k + \omega_k^2\,\tilde{q}_k + \sum_\ell\sum_m\sum_n L_{k\ell mn}\,\tilde{q}_\ell\,\tilde{q}_m\,\tilde{q}_n = P_{0k}\cos\Omega\tau \tag{18}$$

where the variable q_{mn} have been redefined as

$$\{\tilde{q}_k\}^T = \{q_{11}, q_{21},\ldots, q_{N1},\ldots, q_{MN}\}^T \tag{19a}$$

$$c_k = (C_{kk}^{pp}/M_{kk}^{pp}),\; \omega_k^2 = (K_{kk}^{pp}/M_{kk}^{pp}),$$
$$L_{k\ell mn} = (B_{k\ell mn}^{jqrs}/M_{kk}^{pp}),\; P_{0k} = (P_{kp}/M_{kk}^{pp}) \tag{19b}$$

where c_k in terms of damping ratio is $c_k = 2\xi_k\,\omega_k$.

Multimode Analysis Using the Method of Multiple Scales

The method of multiple scales for single-mode solutions of an orthotropic, rectangular panel subjected to harmonic excitations has been presented and applied to simply supported and clamped panels in Ref. 26. However, often it is inadequate to consider only a single-mode analysis. Hence, this section is devoted to a multimode solution of a simply supported panel subjected to uniform harmonic loadings. First, the general multimode solution of the problem using the method of multiple scales is presented. Next, this solution is specialized for two symmetric modes because only symmetric modes are excited for simply supported panels subjected to harmonic excitations.

Now consider the following transformation

$$\tilde{q}_k = \epsilon^{1/2} U_k \tag{20}$$

where ϵ is a small dimensionless parameter. Substituting Eq. (20) into Eq. (18), we obtain

$$\ddot{U}_k + 2\epsilon\mu_k \dot{U}_k + \omega_k^2 U_k + \epsilon \sum_{\ell}\sum_{m}\sum_{n} L_{k\ell mn} U_\ell U_m U_n = F_{0k}\cos\Omega\tau \tag{21}$$

where

$$\mu_k = c_k/2_\epsilon \tag{22}$$

$$F_{0k} = P_{0k}/\epsilon^{1/2} \tag{23}$$

To determine an approximate solution to Eq. (21), the time scales $T_n = \epsilon^n\tau$ are introduced so that the derivatives are transformed as

$$\frac{d}{d\tau} = D_0 + \epsilon D_1 + \dots \tag{24a}$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + \dots \tag{24b}$$

where $D_i = \partial/\partial T_i$.

Following the method of multiple scales, we assume expansions for the U_k in the form

$$U_k(\tau;\epsilon) = U_{k0}(T_0, T_1) + \epsilon U_{k1}(T_0, T_1) + \dots \tag{25}$$

Equation (21) can be solved for the primary resonances as well as the secondary resonances.

Primary Resonance

In general if Ω is near the kth mode frequency, a small-divisor term occurs near this frequency. To aviod this small-divisor term, the forcing amplitude is ordered as $F_{0k} = \epsilon f_k$ and the closeness of Ω to ω_k is defined as

$$\Omega = \omega_k + \epsilon\sigma_k \tag{26}$$

where σ_k is a detuning parameter.

Substituting Eqs. (24-26) into Eq. (21) and equating the coefficients of ϵ^0 and ϵ, we obtain

$$D_0^2 U_{k0} + \omega_k^2 U_{k0} = 0 \tag{27}$$

$$D_0^2 U_{k1} + \omega_k^2 U_{k1} = -2D_0(D_1 U_{k0} + \mu_k U_{k0}) - \sum_{\ell}\sum_{m}\sum_{n} K_{k\ell mn} U_{\ell 0} U_{m0} U_{n0} + f_k \cos\Omega T_0 \tag{28}$$

in which ΩT_0 and $\cos\Omega T_0$ are written as

$$\Omega T_0 = \omega_k T_0 + \sigma_k T_1 \tag{29a}$$

$$\cos\Omega T_0 = \frac{1}{2} e^{i\omega_k T_0} e^{i\sigma_k T_1} + \text{c.c.} \tag{29b}$$

where c.c. stands for complex conjugate. The solution of Eq. (27) is

$$U_{k0} = A_k(T_1)\, e^{i\omega_k T_0} + \bar{A}_k(T_1)\, e^{-i\omega_k T_0} \tag{30}$$

Substitution of Eqs. (29) and (30) into Eq. (28) yields

$$\begin{aligned} D_0^2 U_{k1} + \omega_k^2 U_{k1} = {} & -2i\omega_k (A_k' + \mu_k A_k)\, e^{i\omega_k T_0} \\ & - \sum_{\ell}\sum_{m}\sum_{n} L_{k\ell mn} [A_\ell A_m A_n\, e^{i(\omega_\ell + \omega_m + \omega_n)T_0} \\ & + A_\ell A_m \bar{A}_n\, e^{i(\omega_\ell + \omega_m - \omega_n)T_0} \\ & + A_\ell \bar{A}_m A_n\, e^{i(\omega_\ell + \omega_n - \omega_m)T_0} + A_\ell \bar{A}_m \bar{A}_n\, e^{i(\omega_\ell - \omega_m - \omega_n)T_0}] \\ & + \tfrac{1}{2} f_k\, e^{i\sigma_k T_1} \cdot e^{i\omega_k T_0} + \text{c.c.} \end{aligned} \tag{31}$$

The terms containing $\exp(i\omega_k T_0)$ in Eq. (31) lead to the secular terms in U_{k1}. For example, secular terms are associated with the combination $u_\ell + \omega_n - \omega_m$ if $\ell = k$ and $n = m$. Moreover, there may be other combinations in which

$$\omega_k \approx \omega_\ell \pm \omega_m \pm \omega_n \tag{32a}$$

$$\omega_k = \omega_\ell \pm \omega_m \pm \omega_n + \epsilon\rho_k \tag{32b}$$

where ρ_k is the detuning parameter for the combination resonances. These combinations that lead to small-divisor terms sometimes are referred to as internal resonances that were first considered by Tezak et al.[27] for parametric excitations of beams. The secular terms and the small-divisor terms must not appear in Eq. (31).

As an example, we consider a simple supported rectangular panel having a deflection function as

$$\bar{w} = \sum_i\sum_j q_{ij}(\tau)\sin(i\pi\eta)\sin(j\pi\eta) \tag{33}$$

It can be easily shown that for the given deflection function the forcing amplitude associated with symmetric-antisymmetric modes (i = odd, j = even) or antisymmetric-antisymmetric modes (i = even, j = even) will vanish. Therefore, only symmetric-symmetric modes will be excited. In this paper, the first two symmetric-symmetric modes are considered, that is, $i = 1$, 3, and $j = 1$ in Eq. (33). Two cases might occur: when $\Omega \approx \omega_1$ or $\Omega \approx \omega_3$ corresponding to $k = 1$ or 3. Next, these cases are studied.

Case of Ω Near ω_1

Here, the closeness of Ω to ω_1 can be defined by using Eq. (26) as

$$\Omega = \omega_1 + \epsilon\sigma_1 \tag{34}$$

It follows from Eq. (31) the the only possibility for the internal resonance (coupling between modes) to occur is that $\omega_3 \approx 3\omega_1$. For a simply supported panel, this does not happen, so there will not be coupling between modes. Considering only the first two symmetric modes, breaking Eq. (31) for these two modes and eliminating the secular terms yields

$$-2i\omega_1(A_1' + \mu_1 A_1) - 3\alpha_1 A_1^2 \bar{A}_1 - 2\alpha_2 A_1 A_3 \bar{A}_3 + \tfrac{1}{2} f_1\, e^{i\sigma_1 T_1} = 0 \tag{35}$$

$$-2i\omega_3(A_3' + \mu_3 A_3) - 2\alpha_3 A_1 \bar{A}_1 A_3 - 3\alpha_4 A_3^2 \bar{A}_3 = 0 \tag{36}$$

where $\alpha_1 = L_{1111}$, $\alpha_2 = L_{1133} + L_{1313} + L_{1331}$, $\alpha_3 = L_{3113} + L_{3131} + L_{3311}$, $\alpha_4 = L_{3333}$. Expressing A_n in polar form as

$$A_n(T_1) = \tfrac{1}{2}\psi_n(T_1)\, e^{i\theta_n T_1} \tag{37}$$

substituting Eq. (37) into Eqs. (35), and separating real and imaginary parts, we obtain

$$-\omega_1(\psi_1' + \mu_1 \psi) + (f_1/2) \sin\gamma_1 = 0 \tag{38}$$

$$\omega_1(\sigma_1 - \gamma_1') \psi_1 - 3/8\, \alpha_1 \psi_1^3 - 1/4\, \alpha_2 \psi_1 \psi_3^2 + 1/2\, f_1 \cos\gamma_1 = 0 \tag{39}$$

$$\omega_3(\psi_3' + \mu_3 \psi_3) = 0 \tag{40}$$

$$\omega_3 \psi_3 \theta_3' - 1/4\, \alpha_3 \psi_1^2 \psi_3 - 3/8\, \alpha_4 \psi_3^3 = 0 \tag{41}$$

where

$$\gamma_1 = \sigma_1 T_1 - \theta_1 \tag{42}$$

it follows from Eq. (40) that

$$\psi_3 = \psi_{30} e^{-\mu_3 T_1} \tag{43}$$

This equation indicates that ψ_3 decays with time, that is, $\psi_3 = 0$ as $T_1 \to \infty$. The steady-state solution ($\psi_1' = \gamma_1' = 0$) is described by

$$-\omega_1 \mu_1 \psi_1 + 1/2\, f_1 \sin\gamma_1 = 0 \tag{44}$$

$$\omega_1 \sigma_1 \psi_1 - 3/8\, \alpha_1 \psi_1^3 + 1/2\, f_1 \cos\gamma_1 = 0 \tag{45}$$

Eliminating γ_1 from Eqs. (44) and (45) leads to the following frequency-response equation

$$\omega_1^2 \mu_1^2 \psi_1^2 + (\omega_1 \sigma_1 - 3/8\, \alpha_1 \psi_1^2)^2 \psi_1^2 = 1/4\, f_1^2 \tag{46}$$

This equation is referred to as the primary frequency response of the first mode. As can be seen in this equation there is no interaction between modes.

We note that

$$\omega_1 T_0 + \theta_1 = \omega_1 T_0 + \sigma_1 T_1 - \gamma = (\omega_1 + \epsilon \sigma_1) T_0 - \gamma_1 \tag{47}$$

and therefore the first approximation of the steady-state solution is

$$U_1 = \psi_1 \cos(\Omega \tau - \gamma_1) + \mathcal{O}(\epsilon) \tag{48}$$

Case of Ω Near ω_3

The closeness of Ω_3 can be defined by using Eq. (26), which gives

$$\Omega = \omega_3 + \epsilon \sigma_3 \tag{49}$$

Substituting Eq. (44) into Eq. (31) , eliminating the secular terms, expressing A_n in polar form, and following the same procedure as explained in the previous case, the steady-state solution has the same form as Eqs. (46),

$$\omega_3^2 \mu_3^2 \psi_3^2 + (\omega_3 \sigma_3 - 3/8\, \alpha_4 \psi_3^2)^2 \psi_3^2 = 1/4\, f_3^2 \tag{50}$$

This equation is the frequency response of the second mode, which does not interact with the first mode. The first approximation in this case is

$$U_3 = \psi_3 \cos(\Omega \tau - \gamma_3) + \mathcal{O}(\epsilon) \tag{51}$$

where

$$\gamma_3 = \sigma_3 T_1 - \theta_3 \tag{52}$$

Superharmonic Resonances ($3\Omega \approx \omega_k$)

The superharmonic resonance of the kth mode can be defined according to

$$3\,\Omega = \omega_k + \epsilon \sigma_k \tag{53}$$

As in the case of single-mode analysis, the forcing amplitude need not be ordered. Now, substituting Eq. (25) into Eq. (21) and equating coefficients of like powers of ϵ yields

$$D_0^2 U_{k0} + \omega_k^2 U_{k0} = F_{0k} \cos\Omega T_0 \tag{54}$$

$$D_0^2 U_{k1} + \omega_k^2 U_{k1} = -2D_0(D_1 U_{k0} + \mu_k U_{k0}) - \sum_{\ell}\sum_{m}\sum_{n} L_{k\ell mn} U_{\ell 0} U_{mn} U_{n0} \tag{55}$$

The general solution to Eq. (54) can be expressed as

$$U_{k0} = A_k e^{i\omega_k T_0} + \bar{A}_k e^{-i\omega_k T_0} + H_h (e^{i\Omega T_0} + e^{-i\Omega T_0}) \tag{56}$$

where H_k is defined as

$$H_k = [F_{0k}/2(\omega_k^2 - \Omega^2)] \tag{57}$$

Substitution of Eq. (57) into Eq. (56) yields

$$\begin{aligned} D_0^2 U_{k1} + \omega_k^2 U_{k1} = & -2i\omega_k(A_k' + \mu_k A_k) e^{i\omega_k T_0} - 2i\mu_k \Omega e^{i\Omega T_0} \\ & - \sum_{\ell}\sum_{m}\sum_{n} L_{k\ell mn} [A_\ell A_m A_n e^{i(\omega_\ell + \omega_m + \omega_n)T_0} \\ & + A_\ell A_m \bar{A}_n e^{i(\omega_\ell + \omega_m - \omega_n)T_0} + A_\ell \bar{A}_m A_n e^{i(\omega_\ell - \omega_m + \omega_n)T_0} \\ & + \bar{A}_\ell A_m A_n e^{i(-\omega_\ell + \omega_m + \omega_n)T_0} + 2 A_\ell H_m H_n e^{i\omega_\ell T_0} \\ & + 2 A_m H_\ell H_n e^{i\omega_m T_0} + 2 A_n H_\ell H_m e^{i\omega_n T_0} \\ & + H_\ell H_m H_n (3 e^{i\Omega T_0} + e^{3i\Omega T_0}) + \bar{A}_\ell \bar{A}_m H_n e^{i(\Omega - \omega_\ell - \omega_m)T_0} \\ & + \bar{A}_\ell \bar{A}_n H_m e^{i(\Omega - \omega_\ell - \omega_n)T_0} + \bar{A}_m \bar{A}_n H_\ell e^{i(\Omega - \omega_m - \omega_n)T_0} \\ & + \mathrm{NST} + \mathrm{c.c}] \end{aligned} \tag{58}$$

where NST stands for nonsecular terms. It is to be noted that the expressions on the right-hand side of Eq. (58) are restricted to secular producing terms for superharmonic and subharmonic resonances.

Again, as in the case of primary resonances, two cases may be considered.

Case: $3\,\Omega \approx \omega_1$

This case corresponds to the superharmonic resonance of the lowest mode. Thus, Eq. (53) becomes

$$3\Omega = \mu_1 + \epsilon \sigma_1 \tag{59}$$

Substituting this equation into Eq. (58), eliminating the secular terms, substituting the polar form of A_n from Eq. (37), and following the same procedure as that of the primary resonance case, we obtain

$$\omega_1 (\psi_1' + \mu_1 \psi_1) + Q_1 \sin\gamma_1 = 0 \tag{60}$$

$$\begin{aligned} \omega_1(\sigma_1 - \gamma_1')\psi_1 - 3/8\,\alpha_1 \psi_1^3 - 3\alpha_1 \psi_1 H_1^2 - \alpha_5 \psi_1 H_1 H_3 \\ - 1/8\, \alpha_2 \psi_1 \psi_3^2 - \alpha_2 \psi_1 H_3^2 - Q_1 \cos\gamma_1 = 0 \end{aligned} \tag{61}$$

$$\psi_3' + \mu_3 \psi_3 = 0 \tag{62}$$

$$\begin{aligned} \omega_3 \theta_3' \psi_3 - 3/8\, \alpha_4 \psi_3^3 - 3\alpha_4 \psi_3 H_3^2 - 1/8\, \alpha_3 \psi_3 \psi_1^2 \\ - \alpha_3 \psi_3 H_1^2 - \alpha_6 H_1 H_3 \psi_3 = 0 \end{aligned} \tag{63}$$

where

$$\gamma_n = \sigma_n T - \theta_n,\ n = 1, 3 \tag{64}$$

It is obvious from Eq. (62) that ψ_3 decays with time, so the steady-state condition will be $\psi_1' = \gamma_1' = 0$, which leads to

$$\omega_1^2 \mu_1^2 \psi_1^2 + (\omega_1 \sigma_1 - 3/8\ \alpha_1 \psi_1^2 - 3\alpha_1 H_1^2 - \alpha_5 H_1 H_3 - \alpha_3 H_3^2)^2 \psi_1^2 = Q_1^2 \tag{65}$$

The first approximation of the superharmonic steady-state solution for the first mode is

$$U = \psi_1 \cos(3\Omega\tau - \gamma_1) + 2H_1 \cos\Omega\tau + \mathcal{O}(\epsilon) \tag{66}$$

where ψ_1 can be obtained by using Eq. (65).

Case: $3\ \Omega \approx \omega_3$

The superharmonic resonance in this case can be defined as

$$3\Omega = \omega_3 + \epsilon\ \sigma_3 \tag{67}$$

It can be easily shown that the superharmonic frequency response in this case will be

$$\omega_3^2 \mu_3^2 \psi_3^2 + (\omega_3 \sigma_3 - 3/8\ \alpha_4 \psi_3^2 - 3\alpha_4 H_3^2 - \alpha_6 H_1 H_3 - \alpha_3 H_1^2)^2 \psi_3^2 = Q_3^2 \tag{68}$$

where

$$Q_3 = \alpha_4 H_3^3 + \alpha_3 H_3 H_1^2 + \alpha_6 H_1 H_3^2 + \alpha_8 H_1^3 \tag{69}$$

and

$$\alpha_8 = L_{3111} \tag{70}$$

The first approximation of the steady-state solution for the third mode is

$$U = \psi_3 \cos(3\Omega\tau - \gamma_3) + 2H_3 \cos\Omega\tau + 0(\epsilon) \tag{71}$$

where ψ_3 can be determined from Eq. (68).

Subharmonic Resonance ($\Omega \approx 3\omega_k$)

In this section, we consider only the subharmonic resonance of the lowest mode $k = 1$. Thus the closeness of Ω to $3\omega_1$ is defined as

$$\Omega = 3\ \omega_1 + \epsilon\ \sigma_1 \tag{72}$$

For a simply supported panel, it can be shown that, for low aspect ratios, $\omega_3 \approx 9\ \omega_1$; therefore,

$$\omega_3 = 9\ \omega_1 + \epsilon\ \tilde{\sigma} \tag{73a}$$

or

$$3\Omega = \omega_3 + \epsilon\ \sigma_3,\ \sigma_3 = \sigma_1 - \tilde{\sigma} \tag{73b}$$

In this case, the terms like $\exp i(\Omega - \omega_i - \omega_j)T_0$ are the secular producing terms. By using Eqs. (72) and (73), eliminating the secular terms from Eq. (58), using the polar form of A_n, separating the real and imaginary parts, and enforcing the steady-state condition $\psi_1' = \psi_3' = \gamma_1' = \gamma_3' = 0$ leads to

$$\psi_1^2\ [9\omega_1^2 \mu_1^2 + (\omega_1 \sigma_1 - 9/8\ \alpha_1 \psi_1^2 - 9\alpha_1 H_1^2 - 3/8\ \alpha_2 \psi_3^2 - 3\ \alpha_2 H_3^2 - 3\ \alpha_5 H_1 H_3)^2] = (9\ R_1^2/16)\ \psi_1^4 \tag{74}$$

$$\psi_3^2\ [\mu_3^2 \omega_3^2 + (\sigma_3 \omega_3 - 3/8\ \alpha_4 \psi_3^2 - 3\alpha_4 H_3^2 - 1/8\ \alpha_3 \psi_1^2 - \alpha_3 H_1^2 - \alpha_6 H_1 H_3)^2] = Q_3^2 \tag{75}$$

It follows from Eq. (74) that one of the possible solutions is $\psi_1 = 0$ and $\psi_3 \neq 0$. IF $\psi_1 \neq 0$ and $\psi_3 \neq 0$, the steady-state solution for this case can be obtained by solving the two coupled equations simultaneously. The solution presented by Eqs. (74) and (75) is for low aspect ratios. For high-aspect ratios ($r > 0.1$), $\omega_3 \neq 9\omega_1$ and the coupling between modes does not occur.

Hence, ψ_3 decays, and the frequency response equation becomes

$$9\omega_1^2 \mu_1^2 + (\omega_1 \sigma_1 - 9/8\ \alpha_1 \psi_1^2 - 9\alpha_1 H_1^2 - 3\alpha_1 H_3^2 - 3\alpha_5 H_1 H_3)^2 = (9/16)\ R_1^2 \psi_1^2 \tag{76}$$

It is to be note that the trivial solution is not included in this equation.

Numerical Results and Discusssion

Forced and damped response of a rectangular orthotropic panel subjected to uniform harmonic excitations is studied by using the method of multiple scales (MMS). All the possible resonances—primary, superharmonic, and subharmonic—are studied. The material considered for all computations is a

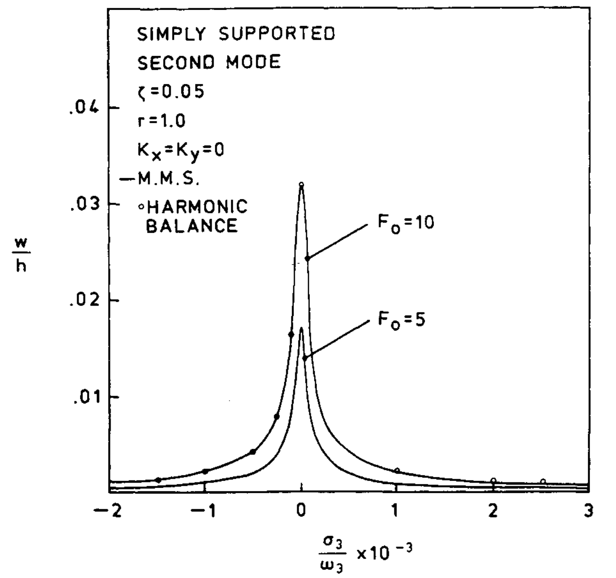

Fig. 1 Frequency response of primary resonance for the second mode and comparison with the harmonic balance method.

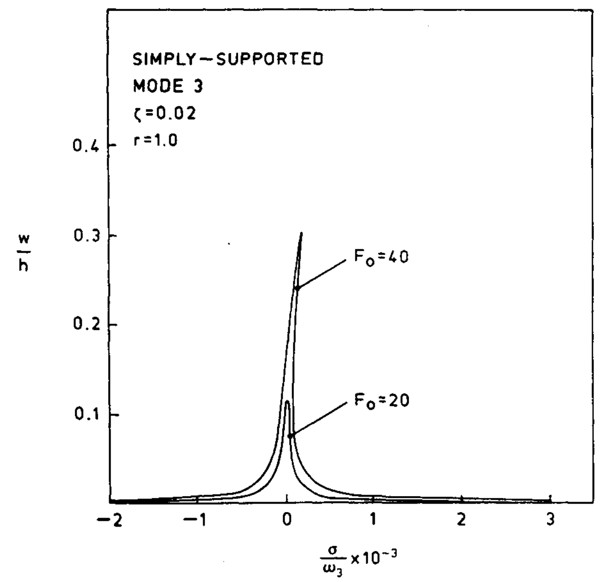

Fig. 2 Frequency response of primary resonance and the effect of higher forcing amplitudes for the second mode.

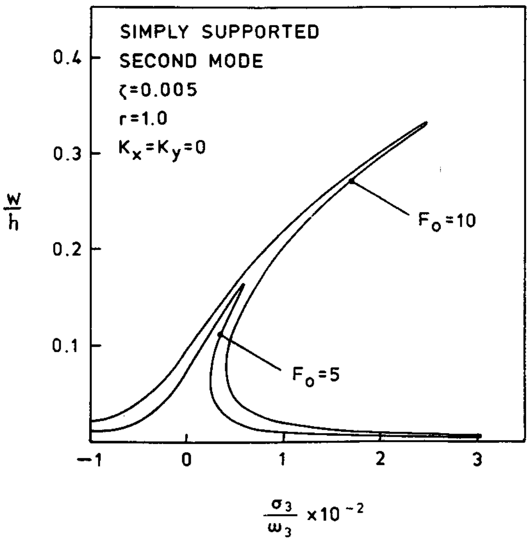

Fig. 3 Effect of lower damping ratio on frequency response of primary resonance for the second mode.

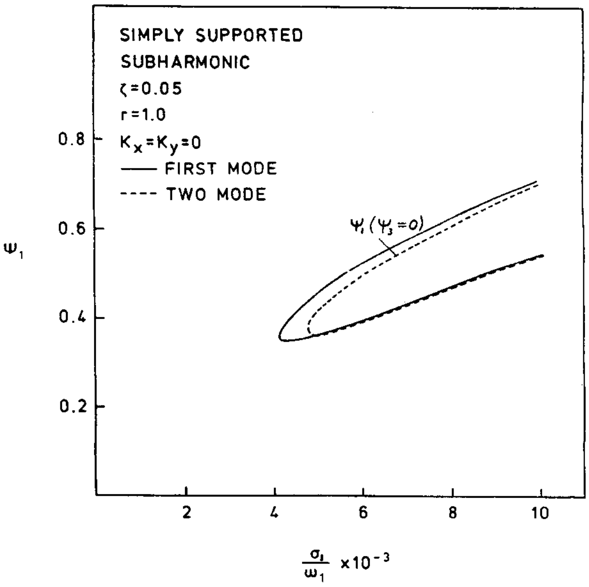

Fig. 6 Subharmonic response of the second mode when $\Omega \approx 3\omega_1$.

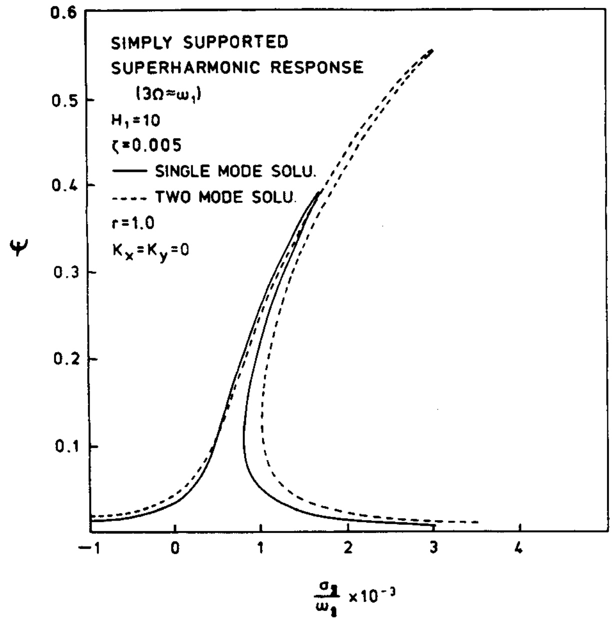

Fig. 4 Contribution of the second mode solution over the single-mode solution for the superharmonic response when $3\Omega \approx \omega_1$.

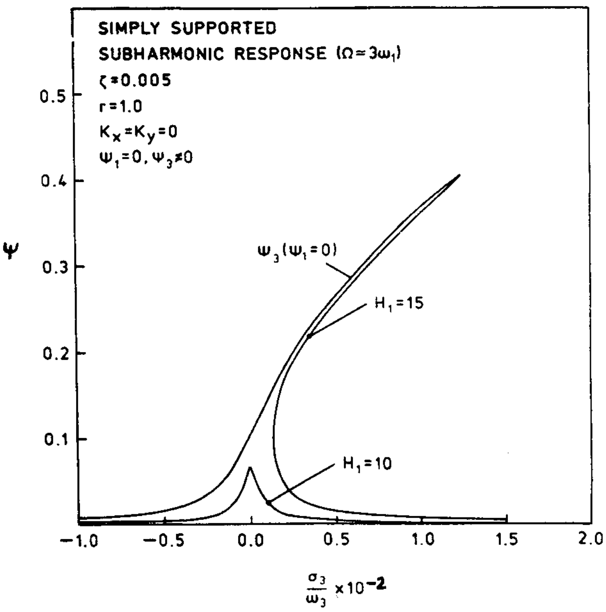

Fig. 7 Contribution of the second-mode solution over the single-mode solution for the superharmonic response.

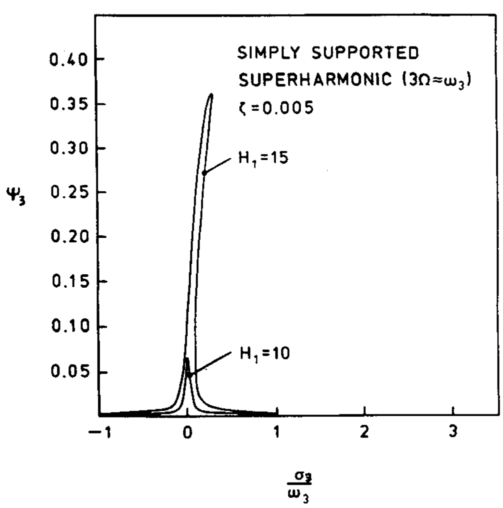

Fig. 5 Supersonic response corresponding to the second mode solution when $3\Omega \approx \omega_3$.

single-layered boron epoxy with the following properties: $E_1 = 17 \times 10^6$ psi, $E_2 = 1.7 \times 10^6$ psi, $G_{12} = 0.68 \times 10^6$, and $\nu_{12} = 0.23$. The value of ϵ for the numerical applications is kept as $\epsilon = 0.0001$.

Figure 1 shows the primary response of the second mode. This figure is the solution to Eq. (50). The comparison between the method of multiple scales and the harmonic balance method shows very good agreement. In this figure, the effect of jump is not visible for the given forcing amplitude, damping ratio, and aspect ratio of 10, 0.05, and 1, respectively. However, by increasing the forcing amplitude or decreasing the damping ratio, the jump phenomena become more visible. These are shown in Figs. 2 and 3. As can be seen in these figures, decreasing the damping ratio is by far more responsible for the jump than increasing the forcing amplitude.

Figure 4 represents the solution to Eq. (65), which shows the contribution of the second-mode solution for the superharmonic response in the neighborhood of $\Omega \approx 1/3\ \omega_1$. This figure shows a higher peak amplitude and the more visible jump phenomenon as compared to the superharmonic response obtained by the first mode only. Figure 5 also shows the super-

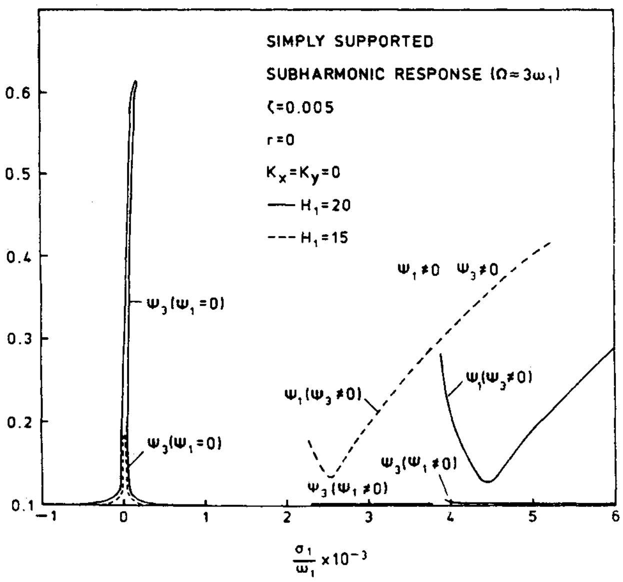

Fig. 8 Subharmonic response of the second mode and the coupling of the modes for a long plate ($\Omega \approx 3\omega_1$).

harmonic response obtained by the second-mode solution in the neighborhood of $\Omega \approx \frac{1}{3}\,\omega_3$. This figure shows a lower peak amplitude for $H_1 = 10$ and $\xi = 0.005$ than that obtained for the case of $\Omega \approx \frac{1}{3}\,\omega_1$. But, by increasing the forcing parameter H_1, the response is significant and should not be ignored. From the superharmonic responses, it also is observed that sharp changes occur due to a small change in forcing amplitude.

Figures 6 and 7 show the subharmonic responses of a square panel ($r = 1$). When $r = 1.0$ or higher, there is no internal resonance and the solution is obtained by Eq. (76), which has the same form as that of the one-mode solution. It contributes only to the response curve. For lower aspect ratios, e.g., $0 \le r \le 0.1$, the internal resonance occurs ($\omega_3 = 9\omega_1 + \epsilon\tilde{\sigma}$) and the solution in this case is obtained by Eqs. (74) and (75), which are coupled and are solved by Newton-Raphson's method. Possible solutions for this case are $\psi_1 = 0$ ($\psi_3 \neq 0$) and $\psi_1 \neq 0$ ($\psi_3 \neq 0$). The effect of coupling comes into play for the latter case that is clearly shown in Fig. 8.

Conclusions

Nonlinear forced oscillation of an orthortropic panel subjected to uniform harmonic excitations is studied. The governing differential equations are derived based on the von Kármán type geometrical nonlinearity. The effect of viscous damping is included in the formulations. Galerkin's method is employed to transform the governing partial differential equations into a set of nonlinear ordinary differential equations. These equations are solved using the method of multiple scales. By using this method, response of the panel for primary, superharmonic, and subharmonic resonances are systematically studied. Solutions obtained by the method of multiple scales are compared with the harmonic balance method for the primary response, and the results are in excellent agreement.

It is concluded that the method of multiple scales is a very powerful analytical tool in dealing with nonlinear forced oscillations of structures. It takes care of damping and gives a clear picture of the panel behavior for all the possible resonances. Finally, this method offers a very systematic procedure to obtain the multimode solution.

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